is a parallelogram. Hence QC is parallel to AR and

$$QC = AR. \tag{2}$$

Now, the triangles PAC, PDB, are similar; for $\angle PCA = \angle PBD$, both standing on the arc AD in a cyclic quadrilateral. Hence

$$AC : DB :: PA : PD. \tag{3}$$

Likewise, the triangles QCA, QDB are similar; and

$$AC : DB :: QC : QD. \tag{4}$$

From (3) and (4),

$$PA : PD :: QC : QD;$$

or

$$PD : QD :: PA : QC;$$

or, by (2),

$$PD : QD :: PA : AR. \tag{5}$$

Again,

$$\angle PDQ = \angle PAR, \tag{6}$$

for

$$\angle PDQ = 180° - \angle ADC = \angle ABC$$

and

$$\angle PAR = \angle ABC.$$

Hence, by (5) and (6) the triangles PDQ and PAR are similar; since they have an angle in each equal and the including sides proportional. Therefore,

$$PA : PD :: PR : PQ;$$

or

$$PA : PD :: 2LN : PQ. \tag{7}$$

By (3) and (7) we have

$$AC : BD :: 2LN : PQ. \tag{8}$$

In a similar way, by drawing QM and prolonging it to S (say), making $MS = QM$, and then drawing PS, we can show that

$$AC : BD :: PQ : 2LM. \tag{9}$$

From (8) and (9),

$$\frac{LN}{PQ} = \frac{1}{2}\frac{AC}{BD}, \qquad \frac{LM}{PQ} = \frac{1}{2}\frac{BD}{AC}.$$

Substituting these values in (1), we have

$$\frac{MN}{PQ} = \frac{1}{2}\left(\frac{AC}{BD} - \frac{BD}{AC}\right); \qquad \text{or} \qquad \frac{MN}{PQ} = \frac{\overline{AC}^2 - \overline{BD}^2}{2AC \cdot BD}.$$

CALCULUS.

333. Proposed by C. N. SCHMALL, New York City.

Evaluate $\int\int\int \sqrt{\frac{1-(x^2+y^2+z^2)}{1+x^2+y^2+z^2}}\, dx\, dy\, dz$, where $x^2 + y^2 + z^2 < 1$.

SOLUTION BY THE PROPOSER.

The given integral evidently depends on the distance of the point (x, y, z) from the origin. As the integration is to be taken for all values of x, y, and z within a sphere of unit radius, center at the origin, the integral may be put into the form

$$u = 4\pi \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}} \cdot r^2 dr,$$

in which the element of volume is a thin spherical shell included between the spheres of radii r, $r + dr$.

Now put $r = \sin\phi$. Then we have

$$u = 4\pi \int_0^{\frac{\pi}{2}} \frac{\cos^2\phi \sin^2\phi}{\sqrt{1+\sin^2\phi}} d\phi = 4\pi \int_0^{\frac{\pi}{2}} \frac{\frac{1}{4}\sin^2 2\phi}{\sqrt{1+\sin^2\phi}} d\phi \tag{1}$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \left(\frac{\frac{1}{4}\sin^2 2\phi}{\sqrt{1+\sin^2\phi}} + \cos 2\phi \sqrt{1+\sin^2\phi} \right) d\phi - 4\pi \int_0^{\frac{\pi}{2}} \cos 2\phi \sqrt{1+\sin^2\phi}\, d\phi$$

$$= 4\pi \Big[\tfrac{1}{2} \sin 2\phi \sqrt{1+\sin^2\phi} \Big]_0^{\frac{\pi}{2}} - 4\pi \int_0^{\frac{\pi}{2}} \cos 2\phi \sqrt{1+\sin^2\phi}\, d\phi$$

$$= -4\pi \int_0^{\frac{\pi}{2}} \cos 2\phi \sqrt{1+\sin^2\phi}\, d\phi = 4\pi \int_0^{\frac{\pi}{2}} (2\sin^2\phi - 1)\sqrt{1+\sin^2\phi}\, d\phi$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{(1-2\cos^2\phi)(1+\sin^2\phi)\, d\phi}{\sqrt{1+\sin^2\phi}}$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{1 - 2\cos^2\phi + \sin^2\phi - 2\sin^2\phi\cos^2\phi}{\sqrt{1+\sin^2\phi}} d\phi$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{1 - 2\cos^2\phi + \sin^2\phi}{\sqrt{1+\sin^2\phi}} d\phi - 2u \quad \text{(by equation (1) above).}$$

Hence

$$3u = 4\pi \int_0^{\frac{\pi}{2}} \frac{3\sin^2\phi - 1}{\sqrt{1+\sin^2\phi}} d\phi = 4\pi \int_0^{\frac{\pi}{2}} \left(3\sqrt{1+\sin^2\phi} - \frac{4}{\sqrt{1+\sin^2\phi}} \right) d\phi$$

$$= 4\pi \int_0^{\frac{\pi}{2}} 3\sqrt{1+\sin^2\phi}\, d\phi - 4\pi \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1+\sin^2\phi}}.$$

Putting $\pi/2 - \phi$ for ϕ, we obtain

$$3u = 12\sqrt{2}\cdot\pi \int_0^{\frac{\pi}{2}} \sqrt{1 - \tfrac{1}{2}\sin^2\phi}\, d\phi - 16\sqrt{2}\cdot\pi \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}.$$

These are complete elliptic integrals of the second and first kinds respectively, and can be readily evaluated. See Peirce's *Short Table of Integrals*, pp. 118–119, and Bromwich's *Infinite Series*, p. 162, ex. 6.

Using the notation employed in the former, we have

$$u = \tfrac{1}{3}\{12\pi\sqrt{2}E(\sqrt{\tfrac{1}{2}},\ \phi) - 16\pi\sqrt{2}F(\sqrt{\tfrac{1}{2}},\ \phi)\}.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

188. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Solve the congruence $3x^2 + 4x + 5 \equiv 0$ [mod 20].

SOLUTION BY WALTER C. EELLS, Whitworth College.

By inspection, solutions of the two congruences,

$$3x^2 + 4x + 5 \equiv 0 \pmod{4}, \tag{1}$$

$$3x^2 + 4x + 5 \equiv 0 \pmod{5}, \tag{2}$$

are $x = \pm 1$ for (1), and $x = 0, 2$ for (2).

Then we have the four systems of linear congruences,

$$(3)\ \begin{cases} x \equiv 1 \pmod{4}, \\ x \equiv 0 \pmod{5}, \end{cases} \qquad (4)\ \begin{cases} x \equiv 1 \pmod{4}, \\ x \equiv 2 \pmod{5}, \end{cases}$$

$$(5)\ \begin{cases} x \equiv -1 \pmod{4}, \\ x \equiv 0 \pmod{5}, \end{cases} \qquad (6)\ \begin{cases} x \equiv -1 \pmod{4}, \\ x \equiv 2 \pmod{5}. \end{cases}$$

To solve system (3), substitute $x = 1 + 4y$ from first congruence in second congruence,

$$1 + 4y \equiv 0 \pmod{5};$$

whence

$$y = 1 + 5k,$$

and

$$x = 1 + 4y = 1 + 4(1 + 5k) = 5 + 20k.$$

Hence, $x \equiv 5 \pmod{20}$, yielding one solution of given congruence. Similarly, systems (4), (5), (6) yield $x = 17, 15, 7$, giving altogether four independent roots of the congruence $3x^2 + 4x + 5 \equiv 0 \pmod{20}$.

Also solved by H. C. FEEMSTER, LOUIS CLARK, S. LEFSCHETZ, E. B. ESCOTT.

No solution of 189 has been received.